Defining Spectral Sequences

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This is a case where I dont think it is helpful to present a general definition and prove a bunch of things. It is a technical tool for keeping track of data and I will try to understand it through some examples. I think it is more standard to do things cohomologically so thats what we will do. This will also allow us to investigate the interaction with the cup product.

1 Chain Complexes

Consider a co-chain complex and its coresponding cohomology in each place. We could take the cohomology of the maps here as well however clearly we would get the same thing back.

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \\
C^2 & & 0 \longrightarrow H^2(C^\bullet) \longrightarrow 0 \\
{\scriptstyle d_1}\uparrow & & \\
C^1 & & 0 \longrightarrow H^1(C^\bullet) \longrightarrow 0 \\
{\scriptstyle d_0}\uparrow & & \\
C^0 & & 0 \longrightarrow H^0(C^\bullet) \longrightarrow 0 \\
\uparrow & & \\
\vdots & & \vdots
\end{array}
$$

2 Filtration of a CW Complex

Consider a finite CW complex X then it has a filtration given by its p skeleton

$$\emptyset = X_{-1} \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X$$

then for any p we get a LES for the pair (X_p, X_{p-1}) given by

$$\cdots \to H^{i-1}(X_{p-1}) \xrightarrow{d_{i-1}} H^i(X_p, X_{p-1}) \xrightarrow{h} H^i(X_p) \xrightarrow{f} H^i(X_{p-1}) \to \cdots$$

or we can write it (for reasons that will become clear)

$$\begin{tikzcd}
& & \cdots \arrow[dll] \\
H^{i-1}(X_{p-1}) \arrow[r, "d_i"] & H^i(X_p, X_{p-1}) \arrow[r, "h"] & H^i(X_p) \arrow[dll, "f"'] \\
H^i(X_{p-1}) \arrow[r] & H^{i+1}(X_p, X_{p-1}) \arrow[r] & H^{i+1}(X_p) \arrow[dll] \\
\cdots & &
\end{tikzcd}$$

If we then vary p we get a family of LES

$$\begin{tikzcd}
H^i(X_{p-1}) \arrow[r] & \mathbf{H^{i+1}(X_p, X_{p-1})} \arrow[r] & \cdots & & & & & & \\
H^{i-1}(X_{p-1}) \arrow[r] & \mathbf{H^i(X_p, X_{p-1})} \arrow[r] & H^i(X_p) \arrow[ull] \arrow[r, equal] & H^i(X_p) \arrow[r] & \mathbf{H^{i+1}(X_{p+1}, X_p)} \arrow[r] & \cdots & & & \\
& & \cdots \arrow[ull] & H^{i-1}(X_p) \arrow[r] & \mathbf{H^i(X_{p+1}, X_p)} \arrow[r] & H^i(X_{p+1}) \arrow[ull] \arrow[r, equal] & H^i(X_{p+1}) \arrow[r] & \mathbf{H^{i+1}(X_{p+2}, X_{p+1})} \arrow[r] & \cdots \\
& & & & & \cdots \arrow[ull] & H^{i-1}(X_{p+1}) \arrow[r] & \mathbf{H^i(X_{p+2}, X_{p+1})} \arrow[r] & H^i(X_{p+2}) \arrow[ull] \\
& & & & & & & & \cdots \arrow[ull]
\end{tikzcd}$$

We stitch these together like so

$$\begin{tikzcd}
H^i(X_{p-1}) \arrow[r, cyan] & \mathbf{H^{i+1}(X_p, X_{p-1})} \arrow[r, cyan] & \cdots & & & & \\
H^{i-1}(X_{p-1}) \arrow[r, cyan] & \mathbf{H^i(X_p, X_{p-1})} \arrow[r, cyan] & H^i(X_p) \arrow[ull] \arrow[r, cyan] & \mathbf{H^{i+1}(X_{p+1}, X_p)} \arrow[r, cyan] & \cdots & & \\
& & H^{i-1}(X_p) \arrow[ull] \arrow[r, cyan] & \mathbf{H^i(X_{p+1}, X_p)} \arrow[r, cyan] & H^i(X_{p+1}) \arrow[ull] \arrow[r, cyan] & \mathbf{H^{i+1}(X_{p+2}, X_{p+1})} \arrow[r, cyan] & \cdots \\
& & & & H^{i-1}(X_{p+1}) \arrow[ull] \arrow[r, cyan] & \mathbf{H^i(X_{p+2}, X_{p+1})} \arrow[r, cyan] & H^i(X_{p+2}) \arrow[ull] \\
& & & & & & \cdots \arrow[ull]
\end{tikzcd}$$

Now we have lots of sequences by traversing this zig zag grid in any direction we want. Here I think one needs some trial and error or perhaps genius.Couple of clues, first if we compose the wrong maps we are just going to get zero, namely if we use two maps from the same sequence it is just zero. Next our idea is to find a new sequence that we can take the cohomology of, so we need to find maps that will square to zero. Serres answer was to take the horrozontal sequences and ensure they satisfy the properties of being a chain complex (two maps compse to zero), namely we consider maps between $H^i(X_p, X_{p-1}) \to H^{i+1}(X_{p+1}, X_p)$ given by composing the blue arrows. Thus we have the sequences

$$\cdots \longrightarrow H^{i+1}(X_p, X_{p-1}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{i}(X_p, X_{p-1}) \longrightarrow \mathbf{H^{i+1}(X_{p+1}, X_p)} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathbf{H^{i}(X_{p+1}, X_p)} \longrightarrow \mathbf{H^{i+1}(X_{p+2}, X_{p+1})} \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{i}(X_{p+2}, X_{p+1}) \longrightarrow \cdots$$

There are a bunch of other maps floating around still as above but these are the sequences that we will take the cohomology of to get new groups!

Question. Could we do it again?

Remark. What did we really use to do this? To create this setup we only used a filtration of the topological space X, not that it was the p skeleton in particular.

3 Definition of a Spectral Sequence

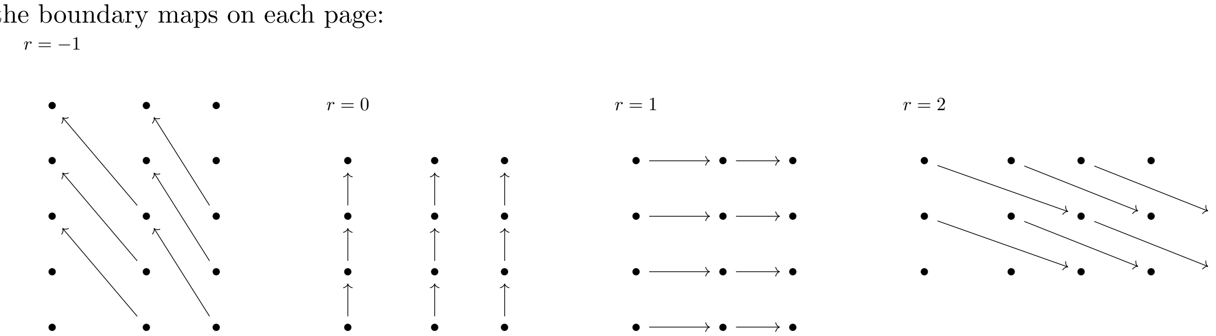
With this in mind we now want to state the definition of a spectral sequence; we already have in mind what we want, a way of taking a family of long exact sequences and cutting them up and putting them back together to take cohomology again and again. The object we will define is the end result of this desired process and then we will be able to prove that some construction or other succeeds in this mission.

The definition of a cohomological (homological) spectral sequence: Let $\mathcal{A}$ be some abelian category for this to take place. Then a spectral sequence is

1. $\{E_r^{pq}\}$ a collection of objects graded by $p, q \in \mathbb{Z}$ and $r \geq a$ for a fixed a.
2. Maps
$$d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}, \quad d^r_{pq} : E_r^{pq} \to E_r^{p-r,q+r-1}$$

such that $d^2 = 0$ and
$$E_{r+1}^{pq} \cong H^*(E_r^{pq}) \quad H_*(E^r_{pq}).$$

The p, q grid for a fixed r is referred to as a page. It is clear that E_{r+1} is a subquotient of E_r. If we place p on the horrizontal axis and q on the verticle axis then we can draw the source and target of the boundary maps on each page:



Importantly each bullet has exactly one incoming and one outgoing map, this allows us to take the cohomology at each point.

Finally we should remark that a *morphism* of spectral sequences is a collection of maps $f^r_{pq} : E^r_{pq} \to F^r_{pq}$ that commutes with d and that the f^{r+1} is the map induced by homology of f^r.

Remark. The above definition is very general and really having a spectral sequence doesnt help you much, of course it can, what you really want is a spectral sequence that has *nice* properties. For example a spectral sequence in which all $E_r^{pq} = 0$ for $p, q \geq 0$ is called 'first quadrant', another property is when the spectral sequence stabilises, that is at some r all further differentials become zero and therefore the groups are forever more fixed, this is refered to as the spectral sequence *degenerating* at page r. If there is only a single non-zero column then we say that the sequence *collapses*.

Remark. Clearly if a sequence degenerates then the page is composed of a family of exact sequences, thus spectral sequences can form a way of proving that certain (families of) sequences are exact.

4 Constructing a SS From an Exact Couple

An exact couple is an algebraic object out of which one can make a spectral sequence. Spectral sequences coming from exact couples will have a couple of nice properties. Notice that if you know a spectral sequence at a given page then there is no way to derive the next page from it, even in principle, the groups are given by the cohomology, but there is no prescription on how to find "good" (non-trivial) boundary maps. For spectral sequences made out of exact couples we will be able to specify just one page and then there is a process by which the others can be generated.

If $\mathcal{A}$ is an abelian category then an exact couple is an exact triangle (the kernel of a map is the image of the previous one) of the form

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

In particular the data of two objects and three maps. This triangle should be thought of as a page of a spectral sequence, where the page is E and the differentials are just jk, we will make this more precise in a moment. The key property of an exact couple is that we can take the so called *derived* couple, which is another exact couple given by

$$\begin{tikzcd} i(D) \arrow[rr, "i"] & & i(D) \arrow[dl, "j'"] \\ & H(E) := \ker(jk)/\mathrm{Im}(jk) \arrow[ul, "k'"] & \end{tikzcd}$$

where j' is defined by $j'(i(d)) = [j(d)]$, $k'[e] = k(e)$. One must check that these maps are well defined and that the resulting triangle is still exact. This should be thought of as the *next page* of the sequence.

If we want to make the relationship to spectral sequences precise we will need to take an exact couple in a bigraded category. Even though we have stated the definitions of spectral sequences for abelian categories lets be real and just think of modules from now on. So consider an exact couple of bi-graded modules

$$\begin{tikzcd} \bigoplus_{p,q} D_{pq} \arrow[rr, "i"] & & \bigoplus_{p,q} D_{pq} \arrow[dl, "j"] \\ & \bigoplus_{p,q} E_{pq} \arrow[ul, "k"] & \end{tikzcd}$$

and moreover we require i to be degree $(1,-1)$, k has degree $-1,0$ and j of $(-0,0)$, to be of degree (c,d) means that $D_{p,q} \to D_{p+c,q+d}$. If we define in the derived couple $D'_{pq} = i(D_{p-1,q+1})$ then these gradings are preserved except for j' which becomes degree $(-1,1)$. Note that we didnt need to restrict the degree of j to zero zero but could have had $(-a,a)$, however we are always able to relable the a variable in our modules (as it doesnt effect the other maps) to set it to 0, this is convenient because then the rth derived couple j will have degree $(r,-r)$. With this setup the E^r_{pq} are the modules in the rth derived couple and the differentials are given by $j^{(r)}k^{(r)}$.

Remark. What we have defined here is a *homological* spectral sequence from an exact couple. The only thing that changes check is the gradings of the maps i,j,k, they become the negatives of what is above!

Remark. Aside from allowing us to specify only one page and generate the others, coming from an exact couple does give some weak convergence properties, see [Wei94, Convergence 5.9.5]

5 Constructing a SS From a Filtration

Consider a filtration of a chain complex

$$\cdots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \cdots$$

Then the inclusion of $F_{p-1}C_\bullet \to F_pC_\bullet$ gives a short exact sequence

$$0 \to F_{p-1}C_\bullet \to F_pC_\bullet \to F_pC_\bullet/F_{p-1}C_\bullet \to 0$$

by relacing the two zeros with the zero map we can roll this up into an exact triangle

$$\begin{tikzcd}
F_{p-1}C_\bullet \arrow[rr, "i"] & & F_pC_\bullet \arrow[dl] \\
& F_pC_\bullet/F_{p-1}C_\bullet \arrow[ul, "0"] &
\end{tikzcd}$$

we can put all the peices of the filtration into an exact triangle of complexes

$$\begin{tikzcd}
\bigoplus F_pC_\bullet \arrow[rr, "i"] & & \bigoplus F_pC_\bullet \arrow[dl] \\
& \bigoplus F_pC_\bullet/F_{p-1}C_\bullet \arrow[ul, "0"] &
\end{tikzcd}$$

where the top row is now of graded degree $+1$. Then the cohomology of this is an exact triangle

$$\begin{tikzcd}
\bigoplus H_{p+q}(F_pC_\bullet) \arrow[rr, "i"] & & \bigoplus H_{p+q}(F_pC_\bullet) \arrow[dl, "j"] \\
& \bigoplus H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet) \arrow[ul, "k"] &
\end{tikzcd}$$

Remark. There is also a more direct way of constructing a spectral sequence from a filtered complex that some people find more intuitive, although it is always the case that we could perform the exact couple construction. Note that the converse is not true, the Bochstein example we will do later does not come from a filtered complex. Im too tired to go through it but it really looks like just explicitly writing out what the above means.

References

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.

Hatcher has a secret chapter 5 on spectral sequences.